

Fuzzy gauge theory and non–locality

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It is argued that the enlargement of the gauge group found in non–commutative gauge theory is more fundamentally thought of as a consequence of the non–locality of the construction and that it was already encountered in an earlier discussion of a non–local gauge theory.

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1. Introduction.

A considerable amount of work has been done on non-commutative gauge theory, stemming, mainly, from its appearance in string theory but also from its formal structure.

One aspect, of many, concerns the burgeoning of the gauge group. By this is meant that the introduced gauge field, for reasons of consistency, cannot be restrained to take values in the Lie algebra, \mathfrak{g} , of the gauge group, G , chosen to act on the matter fields, but must belong to the Lie algebra generated by the independent products of \mathfrak{g} .

A summary of this behaviour is given by Balachandran *et al*, [1], chap.7. Earlier references are Jurčo *et al*, [2], Bonora *et al*, [3], Chachian *et al* [4],

While this ‘problem’ can be alleviated, on the plane (and possibly on the sphere, [5]), using the Seiberg–Witten map, it is still of interest, if only briefly, to examine the mechanism responsible. I will set the scenario in the Moyal plane but any star product would do.

It is, of course, widely recognised, in a general way, that the non-commutative Moyal product is non-local and therefore that due care has to be taken. I take the position that the non-locality is the essential factor. That is, I prefer to say that non-locality implies non-commutativity rather than *vice versa*. I wish to elaborate on this in the present note.

The burgeoning tendency of the gauge group was noticed a long time ago, [6,7], and I wish here to revive this work which was specifically an attempt to extend the gauge principle to a non-local setting. There is not a lot in the present paper that is forward looking.

The formalism developed in [6] and [7] (see also Birch, [8]), is, in some ways, more general than the Moyal situation. The dimension of the manifold in question was not specified although, by default, four-dimensional Lorentzian space-time was implied. In the next section, I recapitulate some of the material in [6] and [7].

2. Non-local gauge theory.

The physical motivation behind the formalism developed in [7] was to see if the possible foam-like structure of space-time (in the terminology of the time) at the Planck length scale had any implications for gauge theory. These might arise because, if points in a (small) region cannot be distinguished, the Yang–Mills

procedure of turning global phase transformations into local ones becomes locally suspect and requires modification. It is difficult to formulate this without using the notion of point and, in [7] the foaminess was replaced by a non-locality, as I now explain.

Formally, I treated space-time coordinates, x , on a par with the weight labels in internal group representations. In [7], for simplicity and explicitness, the internal group was chosen to be $SU(2)$, but one can work generally with G . Let ϕ be a vector in the carrier space of a representation, labelled J , of G (a ‘matter’ field) having dimension $[J]$. Under an infinitesimal gauge transformation I took the change in ϕ at a point to be,

$$\delta\phi_M(x) = i \int_{\mathcal{M}} dx' \Lambda_i(x, x') T^i_M{}^{M'} \phi_{M'}(x'). \quad (1)$$

For simplicity and initially, the integral runs over all of (flat) space-time. Minimal adjustments can be made in the following to turn \mathcal{M} into a Riemannian manifold.

Abstractly, in a compressed notation, (1) is,

$$\delta\phi = i\Lambda\phi \quad (2)$$

where

$$\Lambda = \Lambda_i T^i \quad (3)$$

is an element of the ‘non-local’ Lie algebra. It is represented, in ‘coordinate’ space, by a matrix in the product of space-time and internal space, *i.e.* (3) reads

$$\langle Mx | \Lambda | M'x' \rangle = \langle x | \Lambda_i | x' \rangle \langle M | T^i | M' \rangle \equiv \Lambda_i(x, x') T^i_M{}^{M'}.$$

The T^i are the generators of the Lie algebra, \mathfrak{g} , in the, J representation. As written, they are $[J] \times [J]$ matrices and satisfy

$$[T^i, T^j] = if^{ij}{}_k T^k.$$

I have in mind, initially, $G = SU(n)$ and the T^i ($i = 1 \dots n^2 - 1$) are traceless, hermitian. However, algebraically, I wish to include the unit matrix, which I will denote by T^0 , up to a factor (giving $U(n)$). I therefore set, in (1), $\Lambda_0 = 0$.

Standard, local gauge theory follows on choosing Λ_i of diagonal form,

$$\Lambda_i(x, x') = \Lambda_i(x) \delta(x - x'). \quad (4)$$

Physically one might expect that the off diagonal elements of the general $\Lambda_i(x, x')$ are appreciable only for $|x - x'|$ smaller than some minimum length. The non-local specifications of Λ_i are characteristics of the set-up. Even if the minimum length is very small, *e.g.* the Planck length, the outcome is non-trivial as it involves a matter of principle and not of magnitude.

For the pursuance of the formalism it is not necessary to be more particular and I set up (1) initially in the spirit of ‘suck it and see’.

If the theory is invariant under the local transformation, (4) with $\Lambda_i(x)$ constant (global phase transformations), it will not be so under (1) unless extra fields are introduced. One way of doing this is through the covariant derivative. From (1), the derivative of ϕ changes by

$$\delta \partial_\mu \phi_M(x) = i \int dx' \partial_\mu^x \Lambda_i(x, x') T^i_M{}^{M'} \phi_{M'}(x') \quad (5)$$

or written in the compressed form,

$$\delta P_\mu \phi = i P_\mu \Lambda_i T^i \phi, \quad (6)$$

with the momentum operator, P_μ , where,

$$\langle Mx | P_\mu | M'x' \rangle = i \delta_M^{M'} \partial_\mu \delta(x - x').$$

I rewrite (5) trivially as

$$\delta P_\mu \phi = i [P_\mu, \Lambda_i] T^i \phi + i \Lambda_i T^i P_\mu \phi, \quad (7)$$

so that in the local and constant parameter case, where Λ_i is proportional to $\mathbf{1}$, the unit operator in space-time, the first term in (7) is zero and $\partial_\mu \phi$ transforms like ϕ . If the parameters are functions, local or non-local, this is no longer the case, as remarked, and one seeks for a generalised derivative, which I denote by $K_\mu \phi$, that does transform like ϕ , *i.e.* one insists on,

$$\delta K_\mu \phi = i \Lambda_i T^i K_\mu \phi. \quad (8)$$

K_μ is an operator in both internal space and space-time, as is the gauge potential, A_μ , defined by the split,

$$K_\mu = -i(P_\mu + A_\mu).$$

The general formulae are more neatly expressed in terms of K_μ .

From (8) and (2), I get

$$\delta K_\mu \phi = (\delta K_\mu) \phi + i K_\mu \delta \phi = (\delta K_\mu) \phi + i K_\mu \Lambda_i T^i \phi \equiv i \Lambda_i T^i K_\mu \phi$$

and so, not unexpectedly for a gauge theory,

$$\delta K_\mu = i [\Lambda_i T^i, K_\mu] .$$

In contrast to the usual situation, it is not possible, in general, to expand K_μ linearly in terms of just the generators T^i , for if I do assume,

$$K_\mu = K_{\mu i} T^i , \quad (9)$$

with $K_{\mu 0} = -i P_\mu$, or,

$$\text{tr } A_\mu = 0 \quad i.e. \quad \frac{1}{[J]} \text{tr } K_\mu = -i P_\mu , \quad (10)$$

I find the essential result,

$$\delta K_\mu = \frac{1}{2} [T^i, T^j] \{ \Lambda_i, K_{\mu j} \} + \frac{1}{2} \{ T^i, T^j \} [\Lambda_i, K_{\mu j}]$$

where the curly brackets stand for anti-commutator and tr is an internal trace.

The important point is that the second term here is non-zero, so that δK_μ contains terms quadratic in the generators, T^i , and the new covariant derivative $K_\mu + \delta K_\mu$ is not of the assumed form, (9) with (10). Adding quadratic terms to (9), a repetition of the process will now produce cubic terms, and so on. The procedure will cease, and become consistent, when algebraically dependent products of the T^i are encountered. Essentially by the Cayley–Hamilton theorem the largest independent product has $[J] - 1$ factors. Furthermore, the fact that the trace condition (10) is violated means that the gauge group has enlarged from $\text{SU}(n)$ to $\text{U}([J])$. In [7] the case of $G = \text{SU}(2)$ is considered and this enlargement can be followed in detail. The technique of tensor operators is convenient for the analysis.

The above is exactly the mechanism met with later in fuzzy contexts. Instead of working in coordinate space one can expand the quantities in modes of the Laplacian, say, and then K_μ and Λ_i would be represented by infinite, dual matrices, discrete if \mathcal{M} were compact. Truncating the mode expansions results, crudely, in a fuzzy manifold and thus it is seen that the burgeoning of the gauge group found in this case is but a special example of a more general situation, first outlined in [6,7]. In the next section I show how the Moyal plane fits rather trivially into this scheme.

3. The Moyal plane and non-locality.

Non-locality is introduced into the algebra of functions on the plane through the definition of a star product, an example being the well-known, and much studied, Groenewold–Moyal product,

$$(f * g)(x) = e^{i\frac{1}{2}\theta^{\mu\nu}\partial_\mu^x\partial_\nu^y} f(x)g(y)\big|_{y=x}, \quad \theta^{\mu\nu} = \theta \epsilon^{\mu\nu}. \quad (11)$$

Because of the infinite number of derivatives, this is non-local which can be expressed in various ways, for example *e.g.* Douglas and Nekrasov [9],

$$(f * g)(z) = \int dx \int dy f(x)K(x, y; z)g(y)$$

with

$$K(x, y; z) = \delta(z - x) * \delta(z - y).$$

I will still sometimes refer to Moyal plane as ‘space-time’ and indeed the star product is often applied to four-dimensional space-time in the many phenomenological discussions, involving possibly the standard model. For example, Abel *et al*, [10], analyse the effects of non-commutativity at the Planck scale, $\theta \sim 1/M_P^2$.

The change in a matter field, corresponding to (1) is

$$\delta\phi_M(x) = iT_M^{iM'}(\lambda_i * \phi_{M'})(x) \quad (12)$$

which follows from (1) on making the special choice,

$$\Lambda_i(x, \xi) = (\lambda_i * \delta_\xi)(x) \quad (13)$$

where $\delta_\xi(x) = \delta(x - \xi)$. Abstractly,

$$\Lambda_i = \lambda_i * \mathbf{1}, \quad (14)$$

with $\langle x | \mathbf{1} | y \rangle = \delta(x - y)$.

This is the star product, non-local extension of the local choice, (4) and the burgeoning of the gauge theory noticed in this particular case is simply a consequence of the general result of [6,7] as summarised in section 2.

For pedagogic completeness, I make the identification (13) more explicit in space-time coordinate space by taking ‘matrix elements’ as,

$$\begin{aligned} \langle x | \Lambda_i | \xi \rangle &= \langle x | \lambda_i * \mathbf{1} | \xi \rangle = \int dy \langle x | \lambda_i * | y \rangle \langle y | \mathbf{1} | \xi \rangle \\ &= \int dy \langle x | \lambda_i * | y \rangle \delta(y - \xi). \end{aligned} \quad (15)$$

The Moyal product on the plane corresponds to the choice,

$$\langle x | \lambda_i * | y \rangle = \delta(x - y) \exp \left(-i \frac{1}{2} \theta^{\mu\nu} \overleftrightarrow{\partial}_\nu^y \overrightarrow{\partial}_\mu^x \right) \lambda_i(x),$$

because, when substituted into (15), this gives,

$$\begin{aligned} \langle x | \Lambda_i | \xi \rangle &= \int dy \delta(x - y) \exp \left(-i \frac{1}{2} \theta^{\mu\nu} \overleftrightarrow{\partial}_\nu^y \overrightarrow{\partial}_\mu^x \right) \lambda_i(x) \delta(y - \xi) \\ &= \int dy \delta(x - y) \exp \left(i \frac{1}{2} \theta^{\mu\nu} \overrightarrow{\partial}_\nu^y \overrightarrow{\partial}_\mu^x \right) \lambda_i(x) \delta(y - \xi) \\ &= (\lambda_i * \delta_\xi)(x), \end{aligned}$$

using partial integration and throwing away boundary terms.

Therefore,

$$\begin{aligned} \langle x | \Lambda_i | \phi \rangle &= \int d\xi \langle x | \Lambda_i | \xi \rangle \phi(\xi) \\ &= \int dy \delta(x - y) e^{i \frac{1}{2} \theta^{\mu\nu} \overrightarrow{\partial}_\mu^x \overrightarrow{\partial}_\nu^y} (\lambda_i(x) \phi(y)) \\ &= (\lambda_i * \phi)(x). \end{aligned}$$

as required.

4. Consequences of the formalism and conclusion.

Consistent replacement of the usual by the star product turns ordinary field theory into a non-commutative field theory. In particular, under this replacement, bilinear products in the (integrated) action actually remain unchanged. In the scheme developed in [6,7] this is not so. and a circumstance that must be taken into account in that non-local theory is that quantities constructed from the matter fields, using Clebsch–Gordan techniques such as bilinears and trilinears, which are usually invariant under local G transformations, are no longer so for non-local transformations, (1). For simplicity I look just at the bilinear, I ,

$$I = \psi_M C^{MN} \phi_N = \tilde{\psi} \tilde{C} \phi,$$

where C^{MN} is a G Clebsch–Gordan coefficient which couples the representation to which ψ and ϕ belong, to a scalar. Remember, manifold integrations are implied in this definition.

C is a charge conjugation matrix and, in the usual local case, $\tilde{\psi}\tilde{C}$ transforms contragrediently to ψ *i.e.* like ψ^\dagger .

The change in I under the non-local gauge transformation (1) is (I assume that C is numerically invariant),

$$\begin{aligned}\delta I &= i\tilde{\psi}(\tilde{\Lambda}_i\tilde{T}^i\tilde{C} + \tilde{C}\Lambda_i T^i)\phi \\ &= i\tilde{\psi}\tilde{C}(\Lambda_i - \tilde{\Lambda}_i)T^i\phi\end{aligned}$$

where the transpose is,

$$\langle x | \tilde{\Lambda}_i | x' \rangle = \langle x' | \Lambda_i | x \rangle$$

and I have used the relation

$$C T^i + \tilde{T}^i C = 0, \quad i = 1, \dots$$

Therefore I is not invariant unless Λ_i is symmetric which is not true, in general. (It could be imposed, corresponding to assuming an ‘orthogonal’ transformation.)

The way out of this situation is to allow the ‘metric’ C to vary and become a compensating field, in true gauge theory style, so that I is redefined as,

$$I = \tilde{\psi} \Theta \phi, \tag{16}$$

where Θ is a non-local compensating field which transforms as

$$\delta\Theta = -i(\tilde{T}^i\tilde{\Lambda}_i\Theta + \Theta\Lambda_i T^i) \tag{17}$$

and can be thought of as a metric in the product of G representation space and space-time. If Λ_i is symmetric, from (17) $\delta\Theta$ vanishes if Θ takes the form

$$\langle Mx | \Theta | Ny \rangle = C^{MN} \delta(x - y). \tag{18}$$

For these conditions, Θ is numerically invariant under non-local gauge transformations. By analogy to General Relativity, I say that Θ corresponds to a ‘true’ compensating field if one cannot find a non-local transformation that reduces it to the form (18).

Pursuing this analogy, the ‘Riemannian’ restriction would be to make the covariant derivative of Θ vanish so that raising and lowering commutes with covariant differentiation. This translates to $\tilde{K}_\mu\Theta = -\Theta K_\mu$. But I will not carry on with this line of development nor with the construction of the corresponding curvature, $R_{\mu\nu} = [K_\mu, K_\nu]$. Details can be found in [6,7].

The general notion behind [6,7] was to use the non-local transformations (1) to motivate the introduction of gauge fields and, in an exploratory way, to investigate the structures arising. The $SU(2)$ pion-nucleon system was chosen as a toy model to illustrate the possibilities. The idea was to gauge according to (1) and then take the local limit when relics of the extra non-local fields remain. For example, the bilinear term (16) gave not only the mass, but also the Yukawa pion-nucleon coupling.

Interesting though this result might be, it is incomplete and only speculation. Further, string theory has overtaken events to give the star product version of non-commutativity a more respectable footing.

Incidentally, a few isolated attempts to extend the gauge principle to the non-local case have been published since [6,7] appeared. For the historical connoisseur I give here those of which I am aware, Zupnik, [11], Boiteux and Sobotta, [12] and Dongpei, [13]. Again the non-commutativity is commented upon and also the enlargement of the gauge group.

References.

1. Balachandran, A.P., Kürkcüoğlu, S. and Vaidya, S. *Lectures on Fuzzy SUSY Physics*, arXiv:hep-th/0511114.
2. Jurčo, B., Schraml, S., Schupp, P. and Wess, J. *Eur. Phys. J.* **C17** (2000) 521.
3. Bonora, L., Schnabl, M., Sheikh-Jabbari, M.M. and Tomasiello, A. *Nucl. Phys.* **B589** (2000) 461.
4. Chaichian, M., Presnajder, P., Sheikh-Jabbari, M.M. and Tureanu, A. *Phys. Letts.* **B526** (2002) 132.
5. Grimstrup, J.M., Jonsson, T. and Thorlacius, L, JHEP 01 (2003) 0312 .
6. Birch, A.L. and Dowker, J.S. *J. Phys.* **A2** (1969) 624.
7. Dowker, J.S. *J. Phys.* **A3** (1970) 59.
8. Birch, A.L. Thesis, 1970 (University of Manchester).
9. Douglas, M.R. and Nekrasov, N.A. *Rev. Mod. Phys.* **73** (2001) 977.
10. Abel, S.S., Jaeckel, J., Khoze, V.V. and Ringwald, A. JHEP 09 (2006) 74.
11. Zupnik, B.M. *Letts. Math. Phys* **3** (1979) 543.
12. Boiteux, M. and Sobotta, G. *Lett. al Nuovo Cim.* **40** (1984) 438.
13. Dongpei, Z. *Physica Scripta* **34** (1986) 738.